

# WEIGHTED STRICHARTZ ESTIMATES WITH ANGULAR REGULARITY AND THEIR APPLICATIONS

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**ABSTRACT.** In this paper, we establish an optimal dual version of trace estimate involving angular regularity. Based on this estimate, we get the generalized Morawetz estimates and weighted Strichartz estimates for the solutions to a large class of evolution equations, including the wave and Schrödinger equation. As applications, we prove the Strauss' conjecture with a kind of mild rough data for  $2 \leq n \leq 4$ , and a result of global well-posedness with small data for some nonlinear Schrödinger equation with  $L^2$ -subcritical nonlinearity.

## 1. INTRODUCTION AND MAIN RESULTS

In the analysis of partial differential equations, e.g., the wave and Schrödinger equations, there are many results under the assumption of the spherical symmetry. At the same time, there are many radial estimates dealing with only the radial functions. In general, such estimates or results would not hold for the general case. A natural question is: how much additional angular regularity of the functions is allowed to ensure that the radial results are still valid in that case.

In principle, we believe that, for most of the results with radial assumptions, there are the counterparts for the functions with certain angular regularity. Recently, there have been some interesting results in this direction, see e.g. Machihara-Nakamura-Nakanishi-Ozawa [24], Sterbenz [33], Kato-Nakamura-Ozawa [16], Cho-Ozawa [5]. In this paper, we make an attempt to get some systematic results in this direction.

To this end, we establish an optimal dual version of trace estimate involving angular regularity. Based on this estimate, we get the generalized Morawetz estimates and weighted Strichartz estimates for the solutions to a large class of evolution equations, including the wave and Schrödinger equations. As the applications of our estimates, we prove the Strauss' conjecture with a kind of mild rough data for  $2 \leq n \leq 4$ , and a result of global well-posedness with small data for the nonlinear Schrödinger equation.

Let  $\mathcal{S}$  be the space of Schwartz function,  $\Delta_\omega = \sum_{1 \leq i < j \leq n} \Omega_{ij}^2$  be the Laplace-Beltrami operator on  $S^{n-1} \subset \mathbb{R}^n$  with  $\Omega_{ij} = x_i \partial_j - x_j \partial_i$ ,  $\omega \in S^{n-1}$ ,  $\Lambda_\omega = \sqrt{1 - \Delta_\omega}$ . For  $x \in \mathbb{R}^n$ , we introduce the polar coordinate  $x = r\omega$  with  $r \geq 0$  and  $\omega \in S^{n-1}$ . Let  $\Delta = \sum_{i=1}^n \partial_i^2 = \partial_r^2 + \frac{n-1}{r} \partial_r + \frac{1}{r^2} \Delta_\omega$  be the Laplacian and  $D = \sqrt{-\Delta}$ . Based on the usual Besov spaces  $B_{p,q}^s$ , we introduce the Besov spaces with angular regularity

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as follows ( $m \geq 0$ )

$$B_{p,q,\omega}^{s,m} = \Lambda_\omega^{-m} B_{p,q}^s = \{u \in B_{p,q}^s : \|\Lambda_\omega^m u\|_{B_{p,q}^s} < \infty\}.$$

For the usual Sobolev space  $H^s = B_{2,2}^s$  or the homogeneous Besov space  $\dot{B}_{p,q}^s$ , we can similarly define the spaces  $H_\omega^{s,m} = B_{2,2,\omega}^{s,m}$ ,  $\dot{B}_{p,q,\omega}^{s,m} = \Lambda_\omega^{-m} \dot{B}_{p,q}^s$  and  $\dot{H}_\omega^{s,m} = \dot{B}_{2,2,\omega}^{s,m}$ .

### • Trace Lemma and Sobolev Inequalities with Angular Regularity

We first state the dual version of the trace lemma, which plays the central role in this paper.

**Theorem 1.1** (Dual Version of the Trace Lemma). *Let  $b \in (1, n)$  and  $n \geq 2$ , then we have the following equivalent relation*

$$(1.1) \quad \|\Lambda_\omega^{\frac{b-1}{2}} |x|^{-\frac{b}{2}} \widehat{gd\sigma}(x)\|_{L_x^2} \simeq \|g\|_{L_\omega^2}$$

for any  $g \in L_\omega^2$ .

Here  $A \simeq B$  ( $A \lesssim B$ ) means that  $cB \leq A \leq CB$  ( $A \leq CB$ ) with  $C > c > 0$ , and, in what follows, the constants  $C$  and  $c$  might change at each occurrence.

From (1.1), we know that

$$(1.2) \quad \||x|^{-\frac{b}{2}} \widehat{gd\sigma}(x)\|_{L_x^2} \lesssim \|g\|_{H_\omega^s}$$

if  $s \geq \frac{1-b}{2}$  and  $b \in (1, n)$ .

**Remark 1.1.** *The estimate (1.2) has been proved by Wang in [39] with  $s = 0$  and  $b \in (1, n)$ , and Hoshiro [15] with  $s = \frac{1-b}{2}$  and  $n \geq 3$ . The novelty of our estimate (1.1) consists in getting the equivalent relation instead of the usual inequality.*

**Remark 1.2.** *For the estimate (1.1) or (1.2), the condition on  $b$  is also necessary, see Remark 3.1.*

By duality, scaling and the Sobolev embedding in  $S^{n-1}$

$$H_\omega^s \subset L_\omega^\infty \text{ with } s > \frac{n-1}{2},$$

we can get the following Sobolev estimates which are of independent interest.

**Corollary 1.2.** *Let  $b \in (1, n)$  and  $n \geq 2$ , we have*

$$(1.3) \quad \sup_{r>0} r^{\frac{n-b}{2}} \|f(r\omega)\|_{L_\omega^2} \lesssim \|D^{\frac{b}{2}} \Lambda_\omega^{\frac{1-b}{2}} f\|_{L_x^2}$$

and

$$(1.4) \quad \|D^{-\frac{b}{2}} \Lambda_\omega^{\frac{b-1}{2}} f\|_{L_x^2} \lesssim \| |x|^{\frac{b-n}{2}} f(x) \|_{L_{|x|^{n-1}d|x|}^1 L_\omega^2}$$

for any  $f \in \mathcal{S}$ . Moreover, if  $s > \frac{n-b}{2}$ , then

$$(1.5) \quad \sup_{r>0} r^{\frac{n-b}{2}} \|f(r\omega)\|_{L_\omega^2} \lesssim \|D^{\frac{b}{2}} \Lambda_\omega^s f\|_{L_x^2}.$$

**Remark 1.3.** *When  $f$  is a radial function, the estimate (1.3) has been proved essentially by Li-Zhou (Theorem 2.10 in [22]). The estimate (1.5) for radial  $f$  with  $b = 2$  and  $n \geq 3$  reduces to Ni's inequality [26].*

**Remark 1.4.** Recently, Cho-Ozawa prove the estimate (1.5) with  $s > n - 1 - \frac{b}{2}$  in [5]. Our result improves the angular regularity. Moreover, the requirement of angular regularity in our estimate is essentially optimal, since the index  $\frac{n-b}{2} + \frac{b}{2} = \frac{n}{2}$  is precisely the infimum of  $s$  so that  $H^s \subset L^\infty$ .

As a side remark, we have the following alternative trace estimate for the forbidden case  $b = 1$ .

**Proposition 1.3.** For any compact  $C^\infty$  hypersurface  $M \subset \mathbb{R}^n$  and  $g \in L_M^2$ , we have

$$(1.6) \quad \sup_{x_0, R} R^{-\frac{1}{2}} \|\widehat{gdM}(x)\|_{L_{B(x_0, R)}^2} \lesssim \|g\|_{L_M^2} \lesssim \limsup_{x_0, R \rightarrow \infty} R^{-\frac{1}{2}} \|\widehat{gdM}(x)\|_{L_{B(x_0, R)}^2}.$$

Moreover, by duality, we have  $\dot{B}_{2,1}^{\frac{1}{2}} \subset L_M^2$ , and by rescaling, we get

$$(1.7) \quad r^{\frac{n-1}{2}} \|f(ry)\|_{L_{y \in M}^2} \lesssim \|f\|_{\dot{B}_{2,1}^{\frac{1}{2}}}$$

for any  $f \in \mathcal{S}$ .

**Remark 1.5.** The inequality (1.6) is precisely Theorem 2.1 and Theorem 2.2 of Agmon-Hörmander [1].

Let  $\Delta_M$  be the Laplace-Beltrami operator on  $M$  and  $\Lambda_M = (1 - \Delta_M)^{\frac{1}{2}}$ , then, from (1.7), we can get the following estimate,

$$(1.8) \quad r^{\frac{n-1}{2}} \|f(ry)\|_{L_{y \in M}^\infty} \lesssim \|\Lambda_M^s f\|_{\dot{B}_{2,1}^{\frac{1}{2}}}$$

with  $s > \frac{n-1}{2}$ . This also relaxes the condition  $s > n - \frac{3}{2}$  of Cho-Ozawa [5] in the case of  $M = S^{n-1}$ .

As an application of (1.5) and (1.8), we can get the following result of compact embedding (see e.g. p7 of [5] for the proof, and see e.g. Section 1.7 of Cazenave [3] for the previous radial result)

**Proposition 1.4** (Compact Embedding). *The embedding  $H_{\omega}^{\frac{b}{2}, m} \subset L^p$  is compact for  $b \in (1, n)$ ,  $m > \frac{n-b}{2}$  and  $2 < p < \frac{2n}{n-b}$ . Moreover, the embedding  $B_{2,1,\omega}^{\frac{1}{2}, m} \subset L^p$  is compact for  $m > \frac{n-1}{2}$  and  $2 < p < \frac{2n}{n-1}$ .*

### • Generalized Morawetz Estimates

It is well-known that we can get certain generalized Morawetz inequality (or the local smoothing effect for the dispersive case  $a > 1$ ) from the knowledge of the inequality like (1.2). Here we can give a more refined estimate because of the improved version of the trace lemma.

**Theorem 1.5.** If  $b \in (1, n)$  and  $a > 0$ , then we have

$$(1.9) \quad \| |x|^{-\frac{b}{2}} e^{itD^a} f \|_{L_{t,x}^2} \simeq \| D^{\frac{b-a}{2}} \Lambda_{\omega^{\frac{1-b}{2}}} f \|_{L_x^2}$$

for any  $f \in \mathcal{S}$ . Moreover, for the endpoint case  $b = 1$ , we have the following local estimate

$$(1.10) \quad \sup_{x_0, R} R^{-\frac{1}{2}} \| e^{itD^a} f \|_{L_{t, B(x_0, R)}^2} \lesssim \| D^{\frac{1-a}{2}} f \|_{L_x^2}$$

for any  $f \in \mathcal{S}$ .

The generalized Morawetz estimates usually take the form

$$(1.11) \quad \| |x|^{-\frac{b}{2}} e^{itD^a} f \|_{L_{t,x}^2} \lesssim \| D^{\frac{b-a}{2}} \Lambda_\omega^s f \|_{L_x^2}.$$

The novelty of our estimate (1.9) consists in getting the equivalent relation instead of the usual inequality. Note that a similar phenomenon was found recently by Vega-Visciglia [37] to the estimate (1.10) for the Schrödinger equation ( $a = 2$ ).

**Remark 1.6.** *The estimates (1.11) without angular smoothing index ( $s=0$ ) have been proved by many authors for the special case  $a = 1, 2$ . Morawetz [25] first got the estimate with  $s = 0$ ,  $b = 3$  and  $n \geq 4$  for the wave equation ( $a = 1$ ). Kato and Yajima [18] gave the estimate with  $s = 0$ ,  $b \in (1, 2]$  and  $n \geq 3$  for the Schrödinger equation ( $a = 2$ ). Sugimoto [35] and Vilela [38] prove the estimate for  $a = 2$  with  $s = 0$ . For the estimate with an angular smoothing index, Hoshiro [15] and Sugimoto [35] get some of the estimates for  $a = 1, 2$  with  $s = \frac{1-b}{2}$ .*

**Remark 1.7.** *The local smoothing estimate (1.10) was obtained by Kenig-Ponce-Vega for  $a = 2$  in [21]. As far as we know, the space-localized estimate with  $a = 1$  was first obtained by Smith-Sogge in Lemma 2.2 of [30].*

**Remark 1.8.** *In the case of wave equation ( $a = 1$ ), there also exists an important time-localized estimate, which was first obtained by Keel-Smith-Sogge [19]*

$$\log(2+T)^{-\frac{1}{2}} \| \langle x \rangle^{-\frac{1}{2}} e^{itD} f \|_{L_{[0,T]}^2 L_x^2} \lesssim \| f \|_{L_x^2}.$$

### • Weighted Strichartz Estimates

For the operator  $e^{itD^a}$  with  $a > 0$ , the so-called Strichartz estimates usually take the form (see e.g. Keel-Tao [20])

$$(1.12) \quad \| e^{itD^a} f \|_{L_t^q L_x^r} \lesssim \| f \|_{\dot{H}_x^s},$$

where  $s = \frac{n}{2} - \frac{a}{q} - \frac{n}{r}$  by scaling. The Strichartz estimates have been proved to be very useful in the study of the well-posed problems, see e.g. [3], [23], [29] and [31]. In practice, it is interesting and meaningful to consider the generalized Strichartz estimates of the following type

$$(1.13) \quad \| e^{itD^a} f \|_{L_t^q L_{|x|^{n-1}|d|x|}^r L_\omega^p} \lesssim \| f \|_{\dot{H}_\omega^{s,s_1}}.$$

Moreover, we are interested in obtaining the weighted Strichartz estimates

$$\| |x|^{-\alpha} e^{itD^a} f \|_{L_t^q L_{|x|^{n-1}|d|x|}^r L_\omega^p} \lesssim \| f \|_{\dot{H}_\omega^{s,s_1}}.$$

If we interpolate between the estimates (1.9) and the Sobolev inequality (1.3), we can get the following weighted Strichartz estimates.

**Theorem 1.6** (Weighted Strichartz Estimates). *If  $b \in (1, n)$ ,  $a > 0$  and  $r \in [2, \infty]$ , we have*

$$(1.14) \quad \| |x|^{\frac{n}{2} - \frac{n}{r} - \frac{b}{2}} e^{itD^a} f(x) \|_{L_{t,|x|^{n-1}|d|x|}^r L_\omega^p} \lesssim \| D^{\frac{b-a}{2}} \Lambda_\omega^{\frac{1-b}{2}} f \|_{L_x^2},$$

for any  $f \in \mathcal{S}$ .

The estimates stated in Theorem 1.6 is the homogeneous estimates. In practice, it is often important to give the inhomogeneous estimates. By the Christ-Kiselev lemma (Theorem 1.2 in [7]), we can get the inhomogeneous estimates. In conclusion, we have

**Theorem 1.7.** *Let  $q, \tilde{q} \in [2, \infty]$ ,  $\frac{n}{q} - \alpha, \frac{n}{\tilde{q}} - \tilde{\alpha} \in (0, \frac{n-1}{2})$ ,  $s = \frac{n+a}{q} - \frac{n}{2} - \alpha$ ,  $s_1 = \frac{n-1}{2} + \alpha - \frac{n}{q}$  (note that  $s + s_1 = \frac{a}{q} - \frac{1}{2}$ ), and  $\tilde{s}, \tilde{s}_1$  similarly defined. Then we have*

$$(1.15) \quad \| |x|^{-\alpha} D^s \Lambda_\omega^{s_1} e^{itD^a} f(x) \|_{L_{t, |x|^{n-1}d|x|}^q L_\omega^2} \lesssim \|f\|_{L_x^2},$$

and by duality, one can get

$$(1.16) \quad \left\| \int D^s \Lambda_\omega^{s_1} e^{-isD^a} F(s, x) ds \right\|_{L_x^2} \lesssim \| |x|^\alpha F \|_{L_{t, |x|^{n-1}d|x|}^{q'} L_\omega^2}.$$

Moreover, we have the following inhomogeneous estimates

$$(1.17) \quad \left\| \int_0^t D^s \Lambda_\omega^{s_1} e^{i(t-s)D^a} F(s, x) ds \right\|_{L_t^\infty L_x^2} \lesssim \| |x|^\alpha F \|_{L_{t, |x|^{n-1}d|x|}^{q'} L_\omega^2},$$

and

$$(1.18) \quad \left\| \int_0^t |x|^{-\alpha} D^{s+\tilde{s}} \Lambda_\omega^{s_1+\tilde{s}_1} e^{i(t-s)D^a} F(s, x) ds \right\|_{L_{t, |x|^{n-1}d|x|}^q L_\omega^2} \lesssim \| |x|^{\tilde{\alpha}} F \|_{L_{t, |x|^{n-1}d|x|}^{\tilde{q}'} L_\omega^2}$$

with  $q > \tilde{q}'$ .

**Remark 1.9.** Harmse [11] and Oberlin [27] got the inhomogeneous inequality

$$(1.19) \quad \|w\|_{L_{t,x}^q} \lesssim \|F\|_{L_{t,x}^r},$$

for the solution  $w$  of the equation  $(\partial_t^2 - \Delta)w = F$  with data  $(0, 0)$ , if  $\frac{n+1}{r} - \frac{n+1}{q} = 2$  and

$$(1.20) \quad \frac{n+1}{2n} - \frac{2}{n+1} < \frac{1}{q} < \frac{n-1}{2n}.$$

Our estimate (1.18) generalizes the Harmse-Oberlin estimate for  $a = 1$  to the general cases with weights.

In particular, if we choose  $b \in (1, n)$  such that  $\frac{n}{2} - \frac{n}{r} - \frac{b}{2} = 0$  in Theorem 1.6, we can get the following generalized Strichartz estimates with  $q = r$  in presence of angular regularity.

**Corollary 1.8.** *Let  $a > 0$ ,  $r \in (\frac{2n}{n-1}, \infty)$  and  $p \in [2, \infty)$ , we have*

$$(1.21) \quad \|e^{itD^a} f(x)\|_{L_{t, |x|^{n-1}d|x|}^r L_\omega^p} \lesssim \|D^{\frac{n}{2} - \frac{n+a}{r}} \Lambda_\omega^{\frac{n}{r} - \frac{n-1}{p}} f\|_{L_x^2},$$

for any  $f \in \mathcal{S}$ .

### • Generalized Strichartz Estimates for the Wave Equation

In the case of the wave equation ( $a = 1$ ), it is well known that we have the classical Strichartz estimates (1.12) (see [8]) if

$$(1.22) \quad \frac{1}{q} \leq \min\left(\frac{1}{2}, \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right)\right), (q, r) \neq \left(\max\left(2, \frac{4}{n-1}\right), \infty\right), (q, r) \neq (\infty, \infty).$$

The result in Corollary 1.8 extends the Strichartz estimates to the case of  $q = r < \infty$  and

$$\frac{1}{q} < (n-1)\left(\frac{1}{2} - \frac{1}{r}\right).$$

Thus it is natural to guess that there is a similar result in the more general case  $q \neq r$ . It is in fact the case at least for  $r = p$  (see Sterbenz [33] for  $n \geq 4$ , and Section 3.3 for the full range  $n \geq 2$ ).

**Theorem 1.9.** *Let  $s = n(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}$ ,*

$$(1.23) \quad s_{kn} = \frac{2}{q} - (n-1)(\frac{1}{2} - \frac{1}{r}),$$

*and*

$$(1.24) \quad \frac{n-1}{2}(\frac{1}{2} - \frac{1}{r}) < \frac{1}{q} < (n-1)(\frac{1}{2} - \frac{1}{r}), \quad q \geq 2,$$

*then we have the estimates*

$$\|e^{itD} f\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{H}_\omega^{s, s_1}}$$

*for any  $s_1 > s_{kn}$ .*

In our previous paper [8], we prove that the endpoint Strichartz estimates

$$\|e^{itD} f\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{H}^s}$$

with  $(q, r) = (4, \infty)$  and  $n = 2$  does not hold in general, and holds for radial functions. As a corollary of Theorem 1.9, we can recover this estimate by interpolating the angular  $L^{4-} L^\infty$  estimate and the classical  $L^{4+} L^\infty$  estimate, if we add certain angular regularity. Thus, by combining it with the  $n = 3$  result of [24], we have the following result.

**Corollary 1.10.** *Let  $n = 2, 3$ , for any  $\epsilon > 0$ , we have*

$$(1.25) \quad \|e^{itD} f(x)\|_{L_t^{\frac{4}{n-1}} L_x^\infty} \lesssim \|f\|_{\dot{H}_\omega^{\frac{n+1}{4}, \epsilon}}.$$

### • Generalized Strichartz Estimates for the Schrödinger Equation

The case  $a = 2$  is just the case of the Schrödinger equation. In this case, recall that we have the classical Strichartz estimates (see e.g. Keel-Tao [20])

$$\|e^{it\Delta} f\|_{L_t^q L_x^r} \lesssim \|f\|_{L^2},$$

if

$$\frac{1}{q} = \frac{n}{2}(\frac{1}{2} - \frac{1}{r}) \leq \frac{1}{2}, (q, r, n) \neq (2, \infty, 2).$$

As in the case of the wave equation (see [8]), we can generalize the estimates to

$$(1.26) \quad \|e^{it\Delta} f\|_{L_t^q L_x^r} \lesssim \|D^{\frac{n}{2} - \frac{2}{q} - \frac{n}{r}} f\|_{L_x^2},$$

for

$$(1.27) \quad \frac{1}{q} \leq \min(\frac{1}{2}, \frac{n}{2}(\frac{1}{2} - \frac{1}{r})), (q, r) \neq (\infty, \infty), (2, \infty).$$

Then, the estimates (1.26) require  $\frac{2(n+2)}{n} < r < \infty$  in the case of  $q = r$ . However, by adding some additional angular regularities, we can relax this restriction to  $r > \frac{2n}{n-1}$  in (1.21). Moreover, by interpolate with the known Strichartz estimates for  $q = r = \frac{2(n+2)}{n}$ , we can improve the estimate (1.21) with  $a = 2$  further.

**Corollary 1.11.** *Let  $n > 2$  and*

$$r \in \left( \frac{2n}{n-1}, \frac{2(n+2)}{n} \right),$$

*we have*

$$(1.28) \quad \|e^{it\Delta} f\|_{L_{t,x}^r} \lesssim \|D^{\frac{n}{2} - \frac{n+2}{r}} \Lambda_{\omega}^{\frac{n-1}{n-2}(\frac{n+2}{r} - \frac{n}{2}) + \epsilon} f\|_{L_x^2},$$

*for any  $\epsilon > 0$  and  $f \in \mathcal{S}$ .*

**Remark 1.10.** *Recently, when the data  $f$  is radial, Shao [28] generalizes the estimates (1.26) to the range of  $2\frac{2n+1}{2n-1} < q = r < 2\frac{n+2}{n}$ .*

In general, we **conjecture** that, the estimate (1.13) with  $a = 2$  and  $s = \frac{n}{2} - \frac{2}{q} - \frac{n}{r}$  holds true with  $s_1 > s_{kn}$  for

$$(1.29) \quad \frac{n}{2} \left( \frac{1}{2} - \frac{1}{r} \right) < \frac{1}{q} < \frac{2n-1}{2} \left( \frac{1}{2} - \frac{1}{r} \right) \text{ and } q \geq 2,$$

where

$$(1.30) \quad s_{kn} = \frac{2}{q} + \frac{2n-1}{r} - \frac{n-1}{p} - \frac{n}{2}.$$

It may be interesting to note that in the case of  $p = r$ ,  $s_{kn} = \frac{2}{q} + \frac{n}{r} - \frac{n}{2}$ , which is just  $-s$ .

Next, for the above estimates, we give two applications.

#### • Applications to the Wave Equation

Let  $x \in \mathbb{R}^n$  with  $n \geq 2$ ,  $F_p(u) = \lambda|u|^p$  ( $\lambda \in \mathbb{R} \setminus \{0\}$ ,  $p > 1$ ),  $s_c = \frac{n}{2} - \frac{2}{p-1}$ ,  $s_{sb} = \frac{1}{2} - \frac{1}{p}$ ,  $p_{conf} = 1 + \frac{4}{n-1}$ ,  $p_h = 1 + \frac{4n}{(n+1)(n-1)}$  and  $p_c$  be the solution of the quadratic equation

$$(n-1)p_c^2 - (n+1)p_c - 2 = 0, \quad p_c > 1.$$

Note that  $p > p_c$  if and only if  $s_c > s_{sb}$ . Consider the following semi-linear wave equations for  $u : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$(1.31) \quad \begin{cases} (\partial_t^2 - \Delta)u = F_p(u) \\ u(0, x) = f, \partial_t u(0, x) = g. \end{cases}$$

Strauss' conjecture asserts that the problem (1.31) has a global solution for  $p > p_c$ , when the initial data  $(f, g)$  is sufficiently small and smooth with compact support. This conjecture was finally completed by Georgiev-Lindblad-Sogge in [9] (see also Tataru [36] for another proof). In [9], the authors raised an interesting problem: "Under what kind of the low regularity assumptions on the data, the Strauss conjecture still holds true?"

When  $p \geq p_{conf}$ , Lindblad-Sogge [23] have succeeded in getting the global well-posed result for (1.31) with small data  $(f, g) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}$ , where  $s_c$  is the minimal regularity assumption. When the data  $(f, g) \in \dot{H}^{s_c} \times \dot{H}^{s_c-1}$  are small and radial, there are results dealing with either  $n \leq 4$  or  $p > p_h$ , see e.g. Sogge [31], Lindblad-Sogge [23] and Hidano [12].

We can apply the above estimates to this problem for the small data  $(f, g) \in \dot{H}_{\omega}^{s_c, s_1} \times \dot{H}_{\omega}^{s_c-1, s_1}$  with some  $s_1$ . Following the arguments in Section 8 of Lindblad-Sogge [23], we will first prove the following

**Theorem 1.12.** *Let  $n \geq 2$ ,  $p_h < p < p_{conf}$  (i.e.,  $\frac{1}{2n} < s_c < \frac{1}{2}$ ) and  $s_1 > \frac{1}{2} - s_c$ . Suppose that*

$$(f, g) \in \dot{H}_\omega^{s_c, s_1} \times \dot{H}_\omega^{s_c-1, s_1}$$

*with small enough norm, then there is a unique global weak solution  $u$  to (1.31) satisfying*

$$u \in C_t \dot{H}_x^{s_c} \cap C_t^1 \dot{H}_x^{s_c} \cap L_{t,x}^q \text{ with } q = \frac{(n+1)(p-1)}{2}.$$

Moreover, we can prove the Strauss' Conjecture with a kind of mild rough data for  $n \leq 4$  in the sense of the following theorem.

**Theorem 1.13.** *Let  $2 \leq n \leq 4$ ,  $p_c < p < p_{conf}$  and  $s_1 = \frac{1}{p-1}$ . Suppose that*

$$(f, g) \in \dot{H}_\omega^{s_c, s_1} \times \dot{H}_\omega^{s_c-1, s_1}$$

*with small enough norm, then there is a unique global weak solution  $u \in C_t \dot{H}_\omega^{s_c, s_1} \cap C_t^1 \dot{H}_\omega^{s_c-1, s_1}$  to (1.31) satisfying*

$$|x|^{-\alpha} u \in L_{t, |x|^{n-1}d|x|}^p H_\omega^{s_2},$$

*for  $\alpha = \frac{n+1}{p} - \frac{2}{p-1}$  and  $s_2 = s_1 + s_c - s_{sb}$ .*

**Remark 1.11.** *At the final stage of preparation, we learned from Professor Sogge that they have independently obtained a related result for the equation on the exterior domain (see Hidano-Metcalfe-Smith-Sogge-Zhou [14]).*

## • Applications to the Schrödinger Equation

Let  $x \in \mathbb{R}^n$  with  $n \geq 1$ ,  $F_p(u) = \lambda|u|^p$  or  $F_p(u) = \lambda|u|^{p-1}u$  ( $\lambda \in \mathbb{R} \setminus \{0\}$ ,  $p > 1$ ),  $s_c = \frac{n}{2} - \frac{2}{p-1}$ ,  $p_{L2} = 1 + \frac{4}{n}$ ,  $p_l = 1 + \sqrt{\frac{2}{n-1}}$ . Consider the following nonlinear Schrödinger equation for  $u : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{C}$ ,

$$(1.32) \quad \begin{cases} (i\partial_t - \Delta)u = F_p(u) \\ u(0, x) = f. \end{cases}$$

Cazenave and Weissler [4] proved that if  $s_c \geq 0$  (i.e.  $p \geq p_{L2}$ ) and  $[s_c] < p-1$  ( $s_c < p$  if  $s_c \in \mathbb{Z}$ ), this problem is global well-posed in  $C_t H_x^{s_c}$  (for  $f \in H_x^{s_c}$  with small enough  $\dot{H}_x^{s_c}$  norm). Moreover, if  $s_c < 0$ , the problem is global well-posed in  $C_t L_x^2$ , and fail to be uniformly well-posed in  $C_t H_x^s$  for any  $s < 0$  (see Birnir-Kenig-Ponce-Svanstedt-Vega [2] or Christ-Colliander-Tao [6]). Recently, by assuming  $f \in \dot{H}^{s_c}$  to be small and radial, Hidano [13] get a global result for some  $L^2$ -subcritical nonlinearity  $1 + \frac{4}{n+1} < p < p_{L2}$  and  $n \geq 3$ . We generalize his result to the angular case as follows.

**Theorem 1.14.** *Let  $3 \leq n \leq 6$ ,  $p_l < p < p_{L2}$  and  $s_1 = \frac{1}{p-1}$ . Suppose that  $f \in \dot{H}_\omega^{s_c, s_1}$  with small enough norm, then there is a unique global weak solution  $u \in C_t \dot{H}_\omega^{s_c, s_1}$  to the equation (1.32), satisfying*

$$|x|^{-\alpha} u \in L_{t, |x|^{n-1}d|x|}^q H_\omega^{s_2}$$

*for  $\alpha = \frac{n+2}{q} - \frac{2}{p-1}$  and  $s_2 = \frac{n-1}{2} + \frac{2}{q} - \frac{1}{p-1}$ , where  $q$  satisfy the restriction*

$$\frac{2}{q} \in \left[\frac{1}{p}, 1\right] \cap \left(\frac{2}{p-1} - \frac{n-1}{2}, \frac{2}{p-1} - \frac{n+1}{2p}\right) \cap \left[\frac{1}{p-1} - \frac{n-1}{2p}, \frac{1}{p-1} - \frac{n-3}{2p}\right).$$



This paper will be organized as follows. In Section 2, we collect some preliminary results concerning on the analysis on the sphere and the knowledge of the Bessel functions. In Section 3, we prove the dual version of the trace lemma (Section 3.1), the generalized Morawetz estimates (Section 3.2) and the generalized Strichartz estimates (Section 3.3), in the presence of angular regularity. In Section 4, we give the first application of the estimates obtained in Section 3 to the Strauss' conjecture with a kind of mild rough data for  $n \leq 4$  or  $p_h < p < p_{conf}$ . In the final Section 5, we give another application to a result of global well-posedness with small data for the nonlinear Schrödinger equation.

## 2. PRELIMINARY

For any  $x \in \mathbb{R}^n$ , we introduce the polar coordinates  $r = |x|$  and  $\omega \in S^{n-1}$  such that  $x = r\omega$ . Let  $\langle x \rangle = \sqrt{1 + |x|^2}$  and  $H(t)$  be the usual Heaviside function ( $H(t) = 1$  if  $t \geq 0$  and  $H(t) = 0$  else). For a set  $E$ , we use  $|E|$  to stand for the measure or cardinality of the set  $E$  depending on the context.

The proof of the trace lemma is based on the expansion of a function defined on the sphere with respect to the spherical harmonics. Here we describe the expansion precisely.

Let  $n \geq 2$ . For any  $k \geq 0$ , we denote by  $\mathcal{H}_k$  the space of spherical harmonics of degree  $k$  on  $S^{n-1}$ , by  $d(k) = \frac{2k+n-2}{k} C_{k-1}^{n+k-3} \simeq \langle k \rangle^{n-2}$  its dimension, and by  $\{Y_{k,1}, \dots, Y_{k,d(k)}\}$  the orthonormal basis of  $\mathcal{H}_k$ . It is well known that  $L^2(S^{n-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$  and that  $F(t, x) = F(t, r\omega)$  has the expansion

$$(2.1) \quad F(t, r\omega) = \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} a_{k,l}(t, r) Y_{k,l}(\omega).$$

By orthogonality, we observe that  $\|F(t, r\cdot)\|_{L^2_\omega} = \|a_{k,l}(t, r)\|_{l^2_{k,l}}$ .

Let  $\Delta_\omega$  be the Laplace-Beltrami operator on  $S^{n-1}$  and  $\Lambda_\omega = \sqrt{1 - \Delta_\omega}$ . Then we have  $\Delta_\omega Y_{k,l} = -k(k+n-2)Y_{k,l}$ . Based on this fact, we naturally introduce the Sobolev space  $H_\omega^s = \Lambda_\omega^{-s} L^2_\omega$  on the sphere  $S^{n-1}$  and we have

$$(2.2) \quad \|F(t, r\cdot)\|_{H_\omega^s} = \|\Lambda_\omega^s F(t, r\omega)\|_{L^2_\omega} \simeq \|\langle k \rangle^s a_{k,l}(t, r)\|_{l^2_{k,l}}.$$

The nonlinear estimates on the sphere, such as the Sobolev embedding, the Leibniz rule and the Moser estimate, easily transfers from the Euclidean case (see e.g. [16]). In particular, we have the following Moser estimate (for the Euclidean case, c.f. Kato [17])

$$(2.3) \quad \|\Lambda_\omega^s F_k(u)\|_{L_\omega^r} \lesssim \|u\|_{L_\omega^q}^{k-1} \|\Lambda_\omega^s u\|_{L_\omega^p}$$

for  $s \in [0, m]$  and  $p, q, r \in (1, \infty)$  with  $\frac{1}{r} = \frac{k-1}{q} + \frac{1}{p}$ , where  $F_k \in C^m$  with

$$F(0) = 0, |\partial^\alpha F(x)| \lesssim |x|^{k-|\alpha|}, 1 \leq |\alpha| \leq m \leq k.$$

The spherical harmonics are closely connected with the special functions such as the Gamma function, Bessel functions and so on. Let

$$\Gamma(s) = \int_0^\infty e^{-r} r^{s-1} dr, s > 0$$

be the Gamma function, the Bessel function of order  $k > -\frac{1}{2}$  is defined by

$$(2.4) \quad J_k(t) = \frac{2^{-k} t^k}{\Gamma(k + \frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^1 e^{its} (1 - s^2)^{k-\frac{1}{2}} ds.$$

For these functions, we have some well-known asymptotics. For the Gamma function, we have the Stirling's formula ([10], p.421)

$$(2.5) \quad \Gamma(t) \simeq \sqrt{2\pi} t^{t-\frac{1}{2}} e^{-t}, \text{ as } t \rightarrow \infty.$$

For the Bessel function, we have

$$(2.6) \quad J_k(t) \simeq \sqrt{\frac{2}{\pi}} t^{-\frac{1}{2}} \cos(t - \frac{2k+1}{4}\pi) \text{ as } t \rightarrow \infty, J_k(t) \simeq 2^{-k} t^k \text{ as } t \rightarrow 0.$$

The following result is proved to be very useful. (Note that in their notation,  $P(x) = |x|^k Y_{k,l}(\omega)$ )

**Lemma 1** (IV. Theorem 3.10 in Stein-Weiss [32]). *Let*

$$(2.7) \quad g_{k,l}(\rho) = (2\pi)^{\frac{n}{2}} i^{-k} \int_0^\infty f_{k,l}(r) \rho^{-\frac{n-2}{2}} J_{k+\frac{n-2}{2}}(r\rho) r^{\frac{n}{2}} dr,$$

*then we have*

$$(2.8) \quad f_{k,l}(\widehat{r}) \widehat{Y_{k,l}(\omega)}(\xi) = g_{k,l}(|\xi|) Y_{k,l}(\frac{\xi}{|\xi|})$$

As a corollary, if we set  $f(r) = \delta(r-1)$ , then we get that

$$(2.9) \quad Y_{k,l}(\widehat{\omega}) d\sigma(\omega)(\xi) = (2\pi)^{\frac{n}{2}} i^{-k} |\xi|^{-\frac{n-2}{2}} J_{k+\frac{n-2}{2}}(|\xi|) Y_{k,l}(\frac{\xi}{|\xi|}).$$

In particular, if  $k = 0$ , then we have the well known

$$(2.10) \quad \widehat{d\sigma}(\xi) = (2\pi)^{\frac{n}{2}} |\xi|^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(|\xi|).$$

We will also need the following special form of the Weber-Schafheitlin integral formula for Bessel functions (p.403 of Watson [40]).

**Lemma 2.** *Let  $\mu, \nu, \lambda \in \mathbb{R}$  such that  $\mu + \nu + 1 > \lambda > 0$ , then we have*

$$(2.11) \quad \int_0^\infty \frac{J_\mu(t) J_\nu(t)}{t^\lambda} dt = \frac{\Gamma(\lambda) \Gamma(\frac{\mu+\nu-\lambda+1}{2})}{2^\lambda \Gamma(\frac{\mu-\nu+\lambda+1}{2}) \Gamma(\frac{\nu-\mu+\lambda+1}{2}) \Gamma(\frac{\mu+\nu+\lambda+1}{2})}.$$

### 3. TRACE LEMMA AND GENERALIZED STRICHARTZ ESTIMATES

In this section, we prove the trace lemma and the generalized Morawetz estimates, in the presence of angular regularity. Moreover, we give the proof of the generalized Strichartz estimates for the wave equation (Theorem 1.9).

**3.1. Dual Verison of the Trace Lemma.** In this subsection, we prove the dual version of the trace lemma stated in Theorem 1.1, which plays the central role in this paper. For convenience, we restate it here.

**Theorem 3.1.** *Let  $b \in (1, n)$  and  $n \geq 2$ , then we have the following equivalent relation*

$$(3.1) \quad \|\Lambda_\omega^{\frac{b-1}{2}} |x|^{-\frac{b}{2}} \widehat{gd\sigma}(x)\|_{L_x^2} \simeq \|g\|_{L_\omega^2}.$$

*for any  $g \in L_\omega^2$ .*

By (3.1), we know that

$$(3.2) \quad \| |x|^{-\frac{b}{2}} \widehat{gd\sigma}(x) \|_{L_x^2} \lesssim \|g\|_{H_\omega^s}$$

if  $s \geq \frac{1-b}{2}$  and  $b \in (1, n)$ .

**Remark 3.1.** For the estimate (3.1) or (3.2), the condition on  $b$  is also necessary. In fact, let  $g = 1$  and recalling (2.10) and the well-known asymptotic of the Bessel function (2.6), we know that we have

$$\| |x|^{-\frac{b}{2}} \widehat{d\sigma}(x) \|_{L_x^2} \simeq \| t^{-\frac{b+n-2}{2}} J_{\frac{n-2}{2}}(t) t^{\frac{n-1}{2}} \|_{L_{t>0}^2} \simeq \| t^{-\frac{b-1}{2}} J_{\frac{n-2}{2}}(t) \|_{L_{t>0}^2} < \infty$$

if and only if  $b \in (1, n)$ .

Now we give the proof of Theorem 3.1.

**Proof.** For any  $g \in L_\omega^2$ , we have the expansion formula with respect to the spherical harmonics

$$g(\omega) = \sum_{k=0}^{\infty} \sum_{l=1}^{d(k)} a_{k,l} Y_{k,l}(\omega).$$

By (2.9), we have

$$\widehat{gd\sigma}(x) = \sum_k \sum_{l=1}^{d(k)} a_{k,l} (2\pi)^{\frac{n}{2}} i^{-k} r^{-\frac{n-2}{2}} J_{k+\frac{n-2}{2}}(r) Y_{k,l}(\omega) := \sum_k \sum_{l=1}^{d(k)} b_{k,l}(r) Y_{k,l}(\omega).$$

Since  $Y_{k,l}$  are ortho-normal bases on  $L_\omega^2$ , we have

$$\begin{aligned} \|\Lambda_\omega^{\frac{b-1}{2}} |x|^{-\frac{b}{2}} \widehat{gd\sigma}(x) \|_{L_x^2}^2 &= \int_0^\infty r^{-b} \|\widehat{gd\sigma}(r\omega)\|_{H_\omega^{\frac{b-1}{2}}}^2 r^{n-1} dr \\ &\simeq \sum_{k,l} \langle k \rangle^{b-1} \int_0^\infty r^{-b} |b_{k,l}(r)|^2 r^{n-1} dr \\ &= \sum_{k,l} (2\pi)^n \langle k \rangle^{b-1} |a_{k,l}|^2 \|J_{k+\frac{n-2}{2}}(r) r^{\frac{1-b}{2}}\|_{L_r^2}^2. \end{aligned}$$

If  $b \in (1, n)$ , we can use the formula (2.11) with  $\mu = \nu = k + \frac{n-2}{2}$  and  $\lambda = b-1$  in the last integral. Using the Stirling's formula (2.5) for large  $t$ , we have

$$\|J_{k+\frac{n-2}{2}}(r) r^{\frac{1-b}{2}}\|_{L_r^2}^2 = \frac{\Gamma(b-1)\Gamma(k+\frac{n-b}{2})}{2^{b-1}\Gamma(\frac{b}{2})^2\Gamma(k+\frac{n-2+b}{2})} \simeq \langle k \rangle^{1-b}.$$

Thus, we get that

$$\|\Lambda_\omega^{\frac{b-1}{2}} |x|^{-\frac{b}{2}} \widehat{gd\sigma}(x) \|_{L_x^2}^2 \simeq \sum_{k,l} |a_{k,l}|^2 = \|g\|_{L_\omega^2}^2.$$

■

**3.2. Generalized Morawetz Estimates.** It is well-known that we can get the generalized Morawetz estimates (or the local smoothing effect for the dispersive case  $a > 1$ ) from the knowledge of the trace lemma (3.2). Here we can give a more refined estimate as stated in Theorem 1.5 because of the improved trace lemma in Section 3.1.

**Proof of Theorem 1.5.** Let  $\xi = \lambda\omega$  with  $\omega \in S^{n-1}$  and  $s = \lambda^a$ , we first formally write  $e^{itD^a}f$  as follows,

$$\begin{aligned}
e^{itD^a}f(x) &= (2\pi)^{-n} \int e^{i(t|\xi|^a + x \cdot \xi)} \hat{f}(\xi) d\xi \\
&= (2\pi)^{-n} \int_{S^{n-1}} \int_0^\infty e^{i(t\lambda^a + \lambda x \cdot \omega)} \hat{f}(\lambda\omega) \lambda^{n-1} d\lambda d\sigma(\omega) \\
&= \frac{1}{(2\pi)^n a} \int_{S^{n-1}} \int_0^\infty e^{i(ts + s^{\frac{1}{a}} x \cdot \omega)} \hat{f}(s^{\frac{1}{a}}\omega) s^{\frac{n}{a}-1} ds d\sigma(\omega) \\
&= \frac{1}{(2\pi)^n a} \int_0^\infty e^{its} (\hat{f}(s^{\frac{1}{a}}\omega) d\sigma(\omega))^\vee (s^{\frac{1}{a}}x) s^{\frac{n}{a}-1} ds \\
&= \frac{1}{(2\pi)^n a} \mathcal{F}_s^{-1} \{ (\hat{f}(s^{\frac{1}{a}}\omega) d\sigma(\omega))^\vee (s^{\frac{1}{a}}x) s^{\frac{n}{a}-1} H(s) \} (t) := (h(\cdot, x))^\vee(t).
\end{aligned}$$

By (3.1) and the above expression, we know that

$$\begin{aligned}
\| |x|^{-\frac{b}{2}} e^{itD^a} f \|_{L_{t,x}^2} &\simeq \| |x|^{-\frac{b}{2}} h(s, x) \|_{L_{s,x}^2} \\
&\simeq \| |x|^{-\frac{b}{2}} (\hat{f}(s^{\frac{1}{a}} \cdot) d\sigma)^\vee(x) s^{\frac{n+b}{2a}-1} \|_{L_{s,x}^2} \\
&\simeq \| \Lambda_{\omega^{\frac{1-b}{2}}} \hat{f}(s^{\frac{1}{a}}\omega) s^{\frac{n+b-2a}{2a}} \|_{L_{s,\omega}^2} \\
&\simeq \| \Lambda_{\omega^{\frac{1-b}{2}}} \hat{f}(s\omega) s^{\frac{n+b-2a}{2}} s^{\frac{a-1}{2}} \|_{L_{s,\omega}^2} \\
&\simeq \| \Lambda_{\omega^{\frac{1-b}{2}}} \hat{f}(\xi) |\xi|^{\frac{b-a}{2}} \|_{L_\xi^2} \simeq \| D^{\frac{b-a}{2}} \Lambda_{\omega^{\frac{1-b}{2}}} f \|_{L_x^2}.
\end{aligned}$$

This is just the required estimate (1.9). Note that in the last equality, we have used the fact that

$$\| \Lambda_\omega^s f \|_{L_x^2} \simeq \| \Lambda_\omega^s \hat{f} \|_{L_x^2}.$$

This result can be easily seen, write  $f = \sum_{k,l} f_{k,l}(r) Y_{k,l}(\omega)$ , we have

$$\hat{f}(\xi) = \sum_{k,l} g_{k,l}(|\xi|) Y_{k,l}(\xi/|\xi|),$$

by (2.8), and then applying (2.2),

$$\begin{aligned}
\| \Lambda_\omega^s f \|_{L_x^2} &\simeq \| \langle k \rangle^s f_{k,l}(|x|) \|_{l_{k,l}^2 L_x^2} \\
&\simeq \| \langle k \rangle^s f_{k,l}(|x|) Y_{k,l}(x/|x|) \|_{l_{k,l}^2 L_x^2} \\
&\simeq \| \langle k \rangle^s g_{k,l}(|\xi|) Y_{k,l}(\xi/|\xi|) \|_{l_{k,l}^2 L_\xi^2} \\
&\simeq \| \langle k \rangle^s g_{k,l}(|\xi|) \|_{l_{k,l}^2 L_\xi^2} \simeq \| \Lambda_\omega^s \hat{f} \|_{L_x^2}.
\end{aligned}$$

If we apply (1.6) instead of (3.1) in the proof, we can get the estimate (1.10) by using a similar argument. More precisely, let  $R_s = s^{1/a} R$ , we have

$$\begin{aligned}
R^{-\frac{1}{2}} \| e^{itD^a} f \|_{L_{t,B(x_0,R)}^2} &\simeq R^{-\frac{1}{2}} \| h(s, x) \|_{L_{B(x_0,R),s}^2} \\
&\simeq R_s^{-\frac{1}{2}} \| (\hat{f}(s^{\frac{1}{a}} \cdot) d\sigma)^\vee(x) s^{\frac{n+1}{2a}-1} \|_{L_{s,B(x_0,R_s)}^2} \\
&\lesssim \| \hat{f}(s^{\frac{1}{a}}\omega) s^{\frac{n+1-2a}{2a}} \|_{L_{s,\omega}^2} \\
&\simeq \| \hat{f}(s\omega) s^{\frac{n+1-2a}{2}} s^{\frac{a-1}{2}} \|_{L_{s,\omega}^2} \\
&\simeq \| \hat{f}(\xi) |\xi|^{\frac{1-a}{2}} \|_{L_\xi^2} \simeq \| D^{\frac{1-a}{2}} f \|_{L_x^2}.
\end{aligned}$$

This completes the proof. ■

**3.3. Generalized Strichartz Estimates for the Wave Equation.** In this subsection, we prove Theorem 1.9, based on Proposition 3.4 and 3.5 in Sterbenz [33].

Let  $(q_0, r_0)$  and  $(q_1, r_1)$  be the endpoints of the classical and generalized Strichartz estimates, i.e.,  $(q_1, r_1) = (2, 2\frac{n-1}{n-2})$  and

$$(q_0, r_0) = \begin{cases} (4, \infty) & n = 2, \\ (2, 2\frac{n-1}{n-3}) & n \geq 3. \end{cases}$$

Set  $q_\eta = 2$  for  $n \geq 3$  and  $r_\eta = \infty$  for  $n = 2$ . We recall first Proposition 3.4 and 3.5 in [33].

**Proposition 3.2.** *Let  $u_{1,N} = e^{\pm itD} u_{1,N}(0)$  be a unit frequency, angular frequency localized (at  $2^N$ ) solution to the homogeneous wave equation  $\square u_{1,N} = 0$ . Then for every  $\eta > 0$ , there is a  $(q_\eta, r_\eta)$  such that  $(q_\eta, r_\eta) \rightarrow (q_1, r_1)$  as  $\eta \rightarrow 0$  and the following estimate holds*

$$(3.3) \quad \|u_{1,N}\|_{L_t^{q_\eta} L_x^{r_\eta}} \leq C_\eta N^{1/2+\eta} \|u_{1,N}(0)\|_{L_x^2}.$$

First, recall that to give the proof of Theorem 1.9, we need only to prove the case  $r < \infty$ , by the argument in [8] (which uses generalized Gagliardo-Nirenberg estimate and  $L_t^q L_x^r$  estimate with  $r < \infty$  to control non-endpoint  $L_t^q L_x^\infty$  norm, and can essentially be viewed as a result of interpolation).

By the Littlewood-Paley theory and the Littlewood-Paley-Stein theory (see Strichartz [34]), to prove Theorem 1.9 with  $r < \infty$ , we need only to prove the following inequalities for  $u_{1,N}$ ,

$$(3.4) \quad \|u_{1,N}\|_{L_t^q L_x^r} \lesssim N^{s_{kn}+2\epsilon} \|u_{1,N}(0)\|_{L_x^2}$$

for any  $\epsilon > 0$ . By interpolating with the trivial estimate  $(q, r) = (\infty, 2)$ , we can further reduce it to the estimate with  $r \in (r_1, r_0)$  and  $q = 2$  for  $n \geq 3$ , or  $q \in (q_1, q_0)$  and  $r = \infty$  for  $n = 2$ .

Recall that for the endpoint  $(q_0, r_0)$ , we have the following estimate (see [20] for  $n \neq 3$  with  $\epsilon = 0$  and [24] for  $n = 3$ )

$$(3.5) \quad \|u_{1,N}\|_{L_t^{q_0} L_x^{r_0}} \lesssim N^\epsilon \|u_{1,N}(0)\|_{L_x^2}$$

for any  $\epsilon > 0$ .

Now we can prove the required estimate by interpolation. Fix the parameter  $\epsilon > 0$ , we choose  $\eta \ll 1$  to be fixed later, such that  $(q_\eta, r_\eta) \in [q_1, q] \times [r_1, r]$ , and then choose  $t_\eta \in [0, 1]$  such that

$$(3.6) \quad \frac{t_\eta}{q_\eta} + \frac{1-t_\eta}{q_0} = \frac{1}{q}, \quad \frac{t_\eta}{r_\eta} + \frac{1-t_\eta}{r_0} = \frac{1}{r}$$

i.e.,

$$t_\eta = \left(\frac{1}{q} - \frac{1}{q_0}\right) / \left(\frac{1}{q_\eta} - \frac{1}{q_0}\right) \text{ for } n = 2 \text{ and } t_\eta = \left(\frac{1}{r} - \frac{1}{r_0}\right) / \left(\frac{1}{r_\eta} - \frac{1}{r_0}\right) \text{ for } n \geq 3.$$

Thus if we have

$$(3.7) \quad \left(\frac{1}{2} + \eta\right)t_\eta \leq s_{kn} + \epsilon,$$

then by (3.3) and (3.5),

$$\begin{aligned} \|u_{1,N}\|_{L_t^q L_x^r} &\lesssim \|u_{1,N}\|_{L_t^{q\eta} L_x^{r\eta}}^{t_\eta} \|u_{1,N}\|_{L_t^{q_0} L_x^{r_0}}^{1-t_\eta} \\ &\lesssim N^{(\frac{1}{2}+\eta)t_\eta+\epsilon(1-t_\eta)} \|u_{1,N}(0)\|_{L_x^2} \\ &\lesssim N^{s_{kn}+2\epsilon} \|u_{1,N}(0)\|_{L_x^2}, \end{aligned}$$

which gives the required estimates (3.4).

Note that if  $n = 2$ , we have

$$\lim_{\eta \rightarrow 0} \left(\frac{1}{2} + \eta\right)t_\eta = \frac{1}{2} \left(\frac{1}{q} - \frac{1}{q_0}\right) / \left(\frac{1}{q_1} - \frac{1}{q_0}\right) = \frac{2}{q} - \frac{1}{2} = s_{kn},$$

and if  $n \geq 3$ ,

$$\lim_{\eta \rightarrow 0} \left(\frac{1}{2} + \eta\right)t_\eta = \frac{1}{2} \left(\frac{1}{r} - \frac{1}{r_0}\right) / \left(\frac{1}{r_1} - \frac{1}{r_0}\right) = (n-1) \left(\frac{1}{r} - \frac{1}{r_0}\right) = s_{kn}.$$

Now we can choose  $\eta$  sufficiently small, such that the estimate (3.7) holds true. This completes the proof of Theorem 1.9.

#### 4. STRAUSS' CONJECTURE WITH ROUGH DATA

Let  $n \geq 2$ ,  $F_p(u) = \lambda|u|^p$  ( $\lambda \in \mathbb{R} \setminus \{0\}$ ,  $p > 1$ ),  $s_c = \frac{n}{2} - \frac{2}{p-1}$ ,  $p_{conf} = 1 + \frac{4}{n-1}$ ,  $p_h = 1 + \frac{4n}{(n+1)(n-1)}$  and  $p_c$  be the solution of the quadratic equation

$$(n-1)p_c^2 - (n+1)p_c - 2 = 0, \quad p_c > 1.$$

In this section, we apply the inequalities obtained in Section 1 and Section 3 to the study of the semi-linear wave equation (1.31) with small data  $(f, g) \in \dot{H}_\omega^{s_c, s_1} \times \dot{H}_\omega^{s_c-1, s_1}$  for some  $s_1$ .

**4.1. Global results for  $p_h < p < p_{conf}$ .** In this subsection, we give the proof of Theorem 1.12, following the arguments in Section 8 of Lindblad-Sogge [23].

For the proof, we will invoke the generalized Strichartz estimates in Theorem 1.9 and the inhomogeneous inequality of Harmse-Oberlin (1.19).

We use the usual contraction argument to give the proof. Let  $q = \frac{(n+1)(p-1)}{2}$  and

$$X = \{u \in C_t \dot{H}_x^{s_c} \cap C_t^1 \dot{H}_x^{s_c-1} \cap L_{t,x}^q : \|u\|_{L_{t,x}^q} \leq C\epsilon\}$$

with  $C > 1$  to be determined, and  $s_{kn} = \frac{1}{2} - s_c$ . Define a map  $T : u \mapsto v$  for  $u \in X$ , such that  $v$  is the solution of the equation

$$(\partial_t^2 - \Delta)v = F_p(u), \quad v(0, x) = f, \quad \partial_t v(0, x) = g.$$

Hereafter, we denote by  $v_{hom}$  and  $v_{inh}$  the homogeneous and inhomogeneous part of  $v$  respectively, i.e.,  $v_{hom} = T(0)$  and  $v_{inh} = v - v_{hom}$ .

Since  $u \in X$ , we have  $F_p(u) \in L_{t,x}^{q/p}$ . Note that  $p_h < p < p_{conf}$  if and only if  $\frac{n-1}{2(n+1)} < \frac{1}{q} < \frac{n-1}{2n}$ , which means that the index pair  $(q, q)$  satisfy (1.24). Since  $\|(f, g)\|_{\dot{H}_\omega^{s_c, s_1} \times \dot{H}_\omega^{s_c-1, s_1}} \leq \epsilon$  with  $s_1 > s_{kn}$ , we get from Theorem 1.9

$$\|v_{hom}\|_{C_t \dot{H}_\omega^{s_c, s_1} \cap L_{t,x}^q} + \|\partial_t v_{hom}\|_{C_t \dot{H}_\omega^{s_c-1, s_1}} \leq C_1 \|(f, g)\|_{\dot{H}_\omega^{s_c, s_1} \times \dot{H}_\omega^{s_c-1, s_1}} \leq \frac{C}{2}\epsilon.$$

Note that  $(q/p)' > 2\frac{n+1}{n-1}$  if and only if  $p < p_{conf}$ , we get that

$$\|v_{inh}\|_{C_t \dot{H}_x^{s_c} \cap C_t^1 \dot{H}_x^{s_c-1} \cap L_{t,x}^q} \leq C_2 \|F_p(u)\|_{L_{t,x}^{q/p}} \leq C_2 |\lambda| (C\epsilon)^p \leq \frac{C}{2}\epsilon$$

by the inhomogeneous version of the classical Strichartz estimates (1.22) and the inhomogeneous estimate (1.19) with  $r = q/p$ , if  $\epsilon$  is sufficiently small.

Similarly, one has

$$\|v_1 - v_2\|_{L_{t,x}^q} \leq C_3 \|F_p(u_1) - F_p(u_2)\|_{L_{t,x}^{q/p}} \leq C_4 (C\epsilon)^{p-1} \|u_1 - u_2\|_{L_{t,x}^q} \leq \frac{1}{2} \|u_1 - u_2\|_{L_{t,x}^q}.$$

Thus we have proved that for  $\epsilon > 0$  small enough,  $T$  is a contraction map on  $X$ , which completes the proof of Theorem 1.12.

**4.2. Strauss' Conjecture with mild rough data for  $n \leq 4$ .** In this subsection, we give the proof of Theorem 1.13 by using the weighted Strichartz estimates (1.14) in Theorem 1.6 and Sobolev inequality (1.4) in Corollary 1.2.

Since the method of proof is just the usual contraction argument (as in subsection 4.1), we need only to give some of the key inequalities here.

First, for  $s_1 = \frac{1}{p-1}$ ,  $s_2 = s_1 + s_c - s_{sb}$ ,  $\alpha = \frac{n+1}{p} - \frac{2}{p-1}$ , and any solution  $u$  of the equation  $(\partial_t^2 - \Delta)u = 0$  with data  $(f, g) \in \dot{H}_\omega^{s_c, s_1} \times \dot{H}_\omega^{s_c-1, s_1}$ , we have  $u \in C_t \dot{H}_\omega^{s_c, s_1} \cap C_t^1 \dot{H}_\omega^{s_c-1, s_1}$  and get from (1.14) that

$$|x|^{-\alpha} u \in L_{t, |x|^{n-1}d|x|}^p H_\omega^{s_2},$$

if  $s_c - s_{sb} \in (0, \frac{n-1}{2})$ , i.e.,  $p > p_c$ .

Since  $n \leq 4$  and  $p > p_c \geq 2$ , we have  $\frac{n-1}{2} < 2 \leq [p]$ , where  $[p]$  stand for the integer part of  $p$ . Note that by the Moser estimate (2.3) and the Sobolev embedding, we get the following estimate on  $S^{n-1}$

$$(4.1) \quad \|F_p(u)\|_{H_\omega^a} \lesssim \|u\|_{H_\omega^b}^p$$

with  $b \geq \frac{p-1}{p} \frac{n-1}{2} + \frac{a}{p}$  if  $0 \leq a < \frac{n-1}{2}$  (and thus  $a \leq [p]$ ). Then by letting  $b = s_2$  and  $a = ps_2 - \frac{n-1}{2}(p-1) = \frac{n-1}{2} - \frac{1}{p-1} < \frac{n-1}{2}$ , we have

$$|x|^{-\alpha p} F_p(u) \in L_{t, |x|^{n-1}d|x|}^1 H_\omega^a.$$

Recall the estimate (1.4), if  $\frac{1}{2} - s_c \in (0, \frac{n-1}{2})$ , i.e.,  $1 + \frac{2}{n-1} < p < p_{conf}$ , then

$$F_p(u) \in L_t^1 \dot{H}_\omega^{s_c-1, a+\frac{1}{2}-s_c}.$$

Note that  $a + \frac{1}{2} - s_c = \frac{1}{p-1} = s_1$ , we have  $F_p(u) \in L_t^1 \dot{H}_\omega^{s_c-1, s_1}$ .

By the classical energy estimate, we can get again  $(v, \partial_t v) \in C_t(\dot{H}_\omega^{s_c, s_1} \times \dot{H}_\omega^{s_c-1, s_1})$  and

$$|x|^{-\alpha} v \in L_{t, |x|^{n-1}d|x|}^p H_\omega^{s_2},$$

if  $(\partial_t^2 - \Delta)v = F_p(u)$  with initial data  $(f, g)$ .

If we define the solution space  $X$  to be

$$X = \{u \in C_t \dot{H}_\omega^{s_c, s_1} \cap C_t^1 \dot{H}_\omega^{s_c-1, s_1} : \| |x|^{-\alpha} u \|_{L_{t, |x|^{n-1}d|x|}^p H_\omega^{s_2}} \leq C\epsilon\},$$

then, at last, combining all above, we can prove that the problem (1.31) with  $p_c < p < p_{conf}$  is global well-posed for small data in the space  $C_t \dot{H}_\omega^{s_c, s_1} \cap C_t^1 \dot{H}_\omega^{s_c-1, s_1}$  with  $s_1 = \frac{1}{p-1}$ . Thus we get Theorem 1.13.

## 5. AN APPLICATION TO THE SCHRÖDINGER EQUATION

In this section, we prove Theorem 1.14. As in Section 4.2, we will only give the key inequalities for the proof.

Let  $n \geq 3$ ,  $1 < p < p_{L2}$ ,  $s_1 = \frac{1}{p-1}$ ,  $\alpha = \frac{n+2}{q} - \frac{2}{p-1}$  and  $q \geq 2$  to be determined later. The solution space is

$$X = \{u \in C_t \dot{H}_\omega^{s_c, s_1} : \| |x|^{-\alpha} u \|_{L_{t, |x|^{n-1}d|x|}^q H_\omega^{s_2}} \leq C\epsilon\},$$

By (1.15), we have  $e^{-it\Delta} f \in X$  with

$$(5.1) \quad s_2 = s_1 + \frac{n-1}{2} + \alpha - \frac{n}{q} = \frac{n-1}{2} + \frac{2}{q} - \frac{1}{p-1},$$

for any  $f \in \dot{H}_\omega^{s_c, s_1}$  with norm  $\leq \epsilon$ , if  $\frac{n}{q} - \alpha = \frac{2}{p-1} - \frac{2}{q} \in (0, \frac{n-1}{2})$ , i.e.,

$$(5.2) \quad \frac{2}{p-1} - \frac{n-1}{2} < \frac{2}{q} < \frac{2}{p-1}.$$

By the Moser estimate (4.1), if  $s_3 = ps_2 - \frac{n-1}{2}(p-1) \in [0, [p]]$  and  $s_3 < \frac{n-1}{2}$ , we have

$$(5.3) \quad |x|^{-\alpha p} F_p(u) \in L_{t, |x|^{n-1}d|x|}^{q/p} H_\omega^{s_3}$$

for any  $u$  such that  $|x|^{-\alpha} u \in L_{t, |x|^{n-1}d|x|}^q H_\omega^{s_2}$ . The restrictions on  $s_2$  and  $s_3$  can be reduced to that on  $q$ , i.e.

$$(5.4) \quad \frac{1}{p-1} - \frac{n-1}{2p} \leq \frac{2}{q} < \frac{1}{p-1} - \frac{n-3}{2p}.$$

For any  $u \in X$ , define  $v = Tu$  to be the solution of the equation  $(i\partial_t - \Delta)v = F_p(u)$  with  $v(0) = 0$ . We want to show that  $v \in X$ . Recall the estimate (1.17), we have  $v \in C_t \dot{H}_\omega^{s_c, s_1}$ , if  $q/p \leq 2$ ,  $s_1 \leq s_3 + \frac{n-1}{2} - \alpha p - \frac{n}{(q/p)'}$  and  $\frac{n}{(q/p)'} + \alpha p \in (0, \frac{n-1}{2})$ , i.e.,

$$(5.5) \quad s_1 \leq s_3 - s_c + \frac{3}{2} - \frac{2p}{q},$$

$$(5.6) \quad \frac{2}{q} \geq \frac{1}{p}, \text{ and } \frac{2}{p-1} - \frac{n}{p} < \frac{2}{q} < \frac{2}{p-1} - \frac{n+1}{2p}.$$

Moreover, by estimate (1.18), we have

$$|x|^{-\alpha} v \in L_{t, |x|^{n-1}d|x|}^q H_\omega^{s_2}$$

if  $s_2 - s_3 \leq (\frac{n-1}{2} + \alpha - \frac{n}{q}) + (\frac{n-1}{2} - \alpha p - \frac{n}{(q/p)'})$ , i.e.,

$$(5.7) \quad s_2 - s_3 \leq (\frac{2}{q} - \frac{1}{2}) + (\frac{2}{(q/p)'} - \frac{1}{2}) = 1 - 2\frac{p-1}{q}.$$

The above information is sufficient for us to prove the Theorem 1.14 by using the contraction argument. Thus to complete the proof, we need only to check the requirement on  $q$ . By the expression of  $s_2$  and  $s_3$ , it is easy to check that the inequality (5.5) and (5.7) hold with equality.

The restrictions on  $q \geq 2$  is just (5.2), (5.4) and (5.6). Recall that we assume  $p < p_{L2} < \frac{2n}{n-1}$ , then

$$\frac{2}{p-1} - \frac{n}{p} < \frac{2}{p-1} - \frac{n-1}{2}.$$



This means that the restrictions on  $q$  is just

$$(5.8) \quad \frac{2}{q} \in [\frac{1}{p}, 1] \cap (\frac{2}{p-1} - \frac{n-1}{2}, \frac{2}{p-1} - \frac{n+1}{2p}) \cap [\frac{1}{p-1} - \frac{n-1}{2p}, \frac{1}{p-1} - \frac{n-3}{2p}).$$

This is our requirement on  $q$ . To complete the proof of Theorem 1.14, we conclude that this set is nonempty if

$$(5.9) \quad p-1 \in (\sqrt{\frac{2}{n-1}}, \frac{4}{n}) \text{ and } n \leq 6,$$

i.e.  $p_l < p < p_{L2}$  and  $n \leq 6$ . This can be directly calculated. The somewhat strange index  $p_l$  comes from the condition

$$\frac{2}{p-1} - \frac{n-1}{2} < \frac{1}{p-1} - \frac{n-3}{2p}.$$

We want

$$\frac{1}{p} < \frac{1}{p-1} - \frac{n-3}{2p} \Leftrightarrow p < 1 + \frac{2}{n-3}$$

is true for any  $p = p_{L2} - \epsilon$  with  $\epsilon$  small enough, which gives us the restriction on the dimension  $n \leq 6$ . This finishes the proof of Theorem 1.14.

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